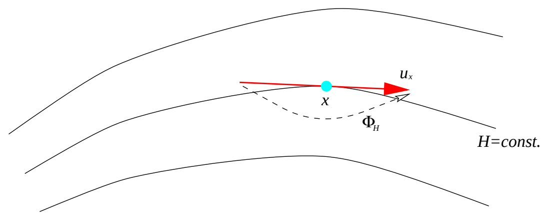
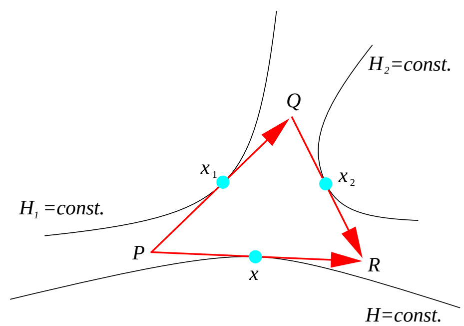
A note on generating functions

Suppose A is an affine symplectic space. There is an affine-invariant view of generating functions of symplectic transformations of A. Namely, let H be a function on A. At any point x we take the vector u_x defined by $d_xH = u_x \lrcorner \omega$ (ω is the symplectic form) and put it in A so that x lies in its middle. Then the map Φ_H sending the tails of u_x's to their heads is a symplectic transformation:



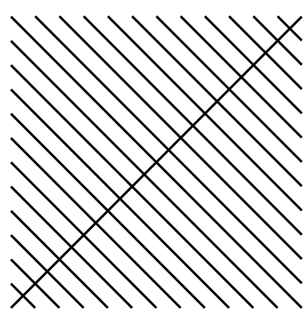
Notice that for infinitesimal H, this is the usual infinitesimal transformation generated by Hamiltonian H. The map $H \mapsto \Phi_H$ is a kind of Cayley transform: choosing an origin in A (to turn it to a vector space) and restricting ourselves to quadratic forms, we get the usual Cayley transform $\mathfrak{sp} \to Sp$.

Symplectic transformations can be composed. The corresponding composition of generating functions is $H(x) = H_1(x_1) + H_2(x_2) +$ symplectic area of $\triangle PQR$:



Recall that the integral kernel of the Moyal product is $K(x_1, x_2, x) = \exp(\sqrt{-1} \times$ symplectic area of $\triangle PQR/\hbar)$. We may notice that $\exp(\sqrt{-1}H/\hbar)$ is the classical part of the Moyal product of $\exp(\sqrt{-1}H_1/\hbar)$ and $\exp(\sqrt{-1}H_2/\hbar)$.

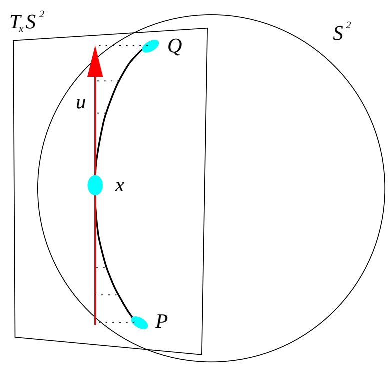
Let us have a look where these claims come from. A symplectic transformation of A is (more-or-less) the same as a Lagrangian submanifold of $\bar{A} \times A$ (the graph of the map). For each point $x \in A$ the symmetry with respect to x is a symplectic map. Identity is also a symplectic map, so that we have many Lagrangian submanifolds of $\bar{A} \times A$:



In this way we have an isomorphism between $\bar{A} \times A$ and T^*A. Explicitely (as one immediately sees from the picture), a pair $(P, Q) \in \bar{A} \times A$ corresponds to $((P + Q)/2, (Q - P) \lrcorner \omega) \in T^*A$. Here the vector-and-its-midpoint picture appears.

Correspondence between generating functions and symplectic transformations is clear now: dH is a Lagrangian submanifold of T^*A, and therefore of $\bar{A} \times A$. Let us also have a look where the composition law comes from. $\bar{A} \times A$ is a symplectic groupoid (the pair groupoid of A). The graph of its multiplication is a Lagrangian submanifold; using the identification of T^*A and $\bar{A} \times A$, it should be given by a closed 1-form on $A \times A \times A$; this 1-form is the differential of the function $(x_1, x_2, x) \mapsto$ symplectic area of $\triangle PQR$. The composition of generating functions and its connection with Moyal product follows.

For the fun of it, let us make a similar construction, replacing A by the sphere S^2 with the area 2-form. Again, symmetry with respect to a point is a symplectic map, therefore we locally have a similar identification between $\overline{S^2} \times S^2$ and T^*S^2; more precisely, there is an isomorphism between the subset of covectors in T^*S^2 of length less than 2 and $\overline{S^2} \times S^2$ with erased pairs of antipodal points. Explicitely, to a non-antipodal pair (P, Q) we associate a point in TS^2 (and thus, via ω, a point in T^*S^2) as on the picture:



x is the midpoint of the shorter geodesic arc PQ and $u \in T_xS^2$ appears by its orthogonal projection. This picture can be derived from the famous theorem of Archimedes, claiming that certain map between cylinder and sphere is area-preserving.

As a result, we have a similar picture of generating functions: for a function H on S^2 and any point x we take the vector u_x defined by $d_xH = u_x \lrcorner \omega$, place it into the tangent plane so that x is in its middle and project it into the sphere; Φ_H maps P to Q. Composition rule looks as before (only triangles are spherical now).

Generally, this picture works with no changes for arbitrary symmetric symplectic space M. Using the symmetries we locally identify $\bar{M} \times M$ with T^*M. Multiplication in this pair groupoid is again given by the symplectic area of a surface bounded by the geodesic triangle PQR with x_1, x_2, x being the midpoints of its sides. The identification between $\bar{M} \times M$ and T^*M is via a projection of M into T_xM, as in the case $M = S^2$: Up to coverings, we embed M into an affine space A. For any $x \in M$, the symmetry with respect to x will be extended to an involution σ_x of A; we project M to T_xM in the direction of A^{σ_x} (the subspace of A fixed by σ_x). Namely, since M is a symmetric space, it is (a covering of) G/G^σ, where G is a Lie group and σ is an involutory automorphism of G. Let $\mathfrak{g} = \mathfrak{g}^\sigma \oplus \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ to ± 1 eigenspaces of $d\sigma$ (to make G/G^σ into a symmetric symplectic space, one has to specify a G^σ-invariant symplectic form on $\mathfrak{p}$). As a homogeneous symplectic space, M can be embedded (up to coverings) into an affine space over $\mathfrak{g}^*$ via (non-equivariant) moment map. If $x \in M$ is fixed by G^σ, T_xM is $\mathfrak{p}^*$ translated to x; we project M to T_xM in the direction of $\mathfrak{g}^{\sigma *}$.

Pavol Ševera, I.H.É.S. (IPDE postdoc)
severa@ihes.fr